[2004] is not listed. Hint to the Mathematical Association of America: The price of Klymchuk [2010] is too high for a 100-page book, and in any case a reprint should cost less than an imported original.

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The Crime of the Century

It is claimed by some that everyone has a secret shame, ranging from a minor faux pas to the tragic. Mine is at the second extreme: Some 35 years ago, *I did poorly in my analysis courses.* This is not something that I can escape! I know that if I run into anyone who knew me then, they will feel it necessary to point out that the funny squiggly symbol is called an "integral sign"; and in any case, transcripts live forever.

However, as I completed my master's degree, my behavior and attitudes changed and I went from being a bad student to a good one. Some of these changes were pathetically overdue (e.g., the recognition that attendance is critical), and other changes might seem trite (I moved from the back of the class to the front). However, my transformation was more qualitative that quantitative. The more-demanding professors now liked me instead of merely tolerating my presence, while the less-demanding ones now tolerated my presence. My grades did not improve much.

Nevertheless, my transformation caused me to believe that each student deserves a second chance and (if needed) a third one. Still, looking back at more than 20 years of full-time teaching, I remember a then-A+ student who reminded me that her first grade from me had been a C in Abstract Algebra I. That was, she said, "a wake-up call." I have given lots of wake-up calls, but she has been the only one to wake up.

You Can Go Home Again

I did only one more semester of pure mathematics after my master's degree, an introductory course in analysis taught very loosely out of Berberian [1970]. The professor gave a small homework assignment at the end of the first lecture; but it was not discussed subsequently, and he never again assigned homework. I alone passed the midterm; there was no final, and we were all given Bs. The lectures never reached the point of defining the Lebesgue integral. The students (and faculty) all knew that this professor could not (or would not) teach, and so did the administration. I believe that his Faustian bargain, presumably to get tenure, was that he would stop flunking students. However, I did not sign up for that course, pay tuition, and attend all of the lectures simply to get a (minimally) passing grade. I was 25, I was now serious, and—since the course seemed representative of that department—I left. That was 35 years ago, but it still makes me angry. You can go home again, but in general you do not get second chances.

When I went into industry, I needed to study probability; and I used that need as an excuse to study analysis. It did not take a long time to acquire most of the analysis that I had failed to learn in my previous courses. I did my Ph.D. in industrial engineering, and it so happens that there are individuals in that field who know analysis. My experience, though, was that professors mentioned "measure" or "Lebesgue" as a shibboleth to show that they were mathematically sophisticated. One professor insisted that a random variable is not a function from a sample space to the reals but is a "measurable" function. This single use of a form of the word "measure," as well as knowledge of such theoretical esoterica as the St. Petersburg paradox, were enough to distinguish him as one of the great theoretical geniuses of the 20th century. Now, I have taught analysis countless times during the last 20 years—but that means nothing. The fact is, that with most classes, if I were to plug a hole in some proof by invoking astrology or the Magna Carta, the students would not object.

Two Books

The two books reviewed here, *Elementary Real Analysis*, 2nd ed. (hereafter, ERA) by Thomson et al., and *Real Analysis*, 2nd ed. (hereafter, RA) by Bruckner et al., are my new favorite books in analysis. They are large inexpensive paperbacks, very well-written and organized—the sort of book that is easy to pick up and start reading in the middle. Also, as texts go, they are fairly complete. The combination of factors makes them great references, but it is also a weakness in them as texts. The authors use a scissors icon to indicate text that can be skipped; however, since most instructors will have to skip more material than the passages marked, skipping large swaths of text can make the going tougher for the student.

These books are friendly, as opposed to dry, but they are too strong for the weakest students. Then again, one can argue that the whole idea of analysis courses is to filter out the weakest students. For example, Bryant [1990] is at the most elementary level. It is well written, sometimes quite ingenious, and makes for a decent text; but it goes to such lengths in helping the student to navigate delta-epsilon arguments that sometimes there is little for the instructor to do. But the more serious problem that recurs in any text that is so friendly is that if the student needs so much hand-holding, then maybe the student is in the wrong field. Bryant's text is ideal for self-study by the sophomore or junior trying to get a head start on analysis, but it may be too elementary for a course text.

The Question of Applications

At *The UMAP Journal*, we are interested in applied mathematics and especially in modeling. Analysis undergirds much applied mathematics. There are texts that try to combine analysis and applications; Cooper [2004] is one such book and it received a rave review from Steven Krantz [2005]. Estep [2002] is a strictly undergraduate book that is also very strong on applications, and I like it a great deal. Estep spends a great deal of time on Lipschitz-continuous functions, which leads to a nice analysis of fixed-point iteration. Uniform continuity then appears as a generalization of Lipschitz continuity; this is pedagogically a very nice way of teaching analysis. However, the book is 621 pp long. In a pure analysis course, I simply do not have time for topics such as fixed-point iteration. The question then is, what course do you use this book or Cooper's book for?

A Model Text

Understanding Analysis by Abbott [2001] is one of the most-used analysis texts in 2010. It is the text that I am teaching from, and I consider it a model undergraduate text in analysis. At 257 pp, it is less than half the length of ERA but at the same depth. ERA simply covers more material in more detail than Abbot.

ERA (and RA) are pedagogically strong; they have good examples; their exercises are excellent, and the writing is lucid and informative. They simply are more complete than many other texts. I would recommend RA to any student studying for a Ph.D. qualifying exam in analysis. (The other book that I would recommend would be *Lebesgue Integration on Euclidean Space* by Frank Jones [2001]. Despite its title, it might be a great first resource for students encountering Lebesgue integration. It has 588 pp but is quite readable.) I am not much impressed by most of the short books intended to be quick introductions to Lebesgue theory. Lebesgue integration has many details, and these details cannot be skipped—which is a reason to go to the generalized Riemann integral.

Riemann, Lebesgue, or Riemann Part II (the Revenge)?

ERA, in my view, is virtually a complete course in undergraduate real analysis; and similarly, RA is is a complete course in graduate real analysis. As a result, ERA covers the Riemann integral and RA covers the Lebesgue integral.

Jean Dieudonné suggested scrapping the Riemann integral in favor of other integrals (I myself heard him say this). In fact, there is much discussion on discarding the Riemann integral by the authors of the books reviewed here, at their Website `http://www.classicalrealanalysis.com`.

The standard higher integral is the Lebesgue integral. Its advantages over the Riemann integral are as follows:

- The Lebesgue integral applies to every function that the Riemann integral handles and then some more. The simplest example of a function that the Lebesgue integral will handle that the Riemann integral will not handle is probably the indicator function of the rational numbers, a historically important function attributed to Dirichlet. However, this function holds no interest for applied mathematicians, engineers, and physicists. It does have an interesting interpretation in probability, viz., if one does an infinite number of tosses of a fair coin, the probability that the sequence of heads and tails will eventually fall into a repeating pattern is zero. But this result too is outside of applied mathematics (to the extent that applied mathematics is about the real world). This particular view of the Lebesgue measure of subsets of $[0, 1)$—in terms of coin tossing sequences—is in fact the opening motivation for Lebesgue theory in Adams and Guillemin [1996]. I consider this book along with Capiński and Kopp [1999] as among the best short introductions to the Lebesgue integral. (However, one should go into both with a prior knowledge of probability). The standard reference for Lebesgue theory and probability is Billingsley [1995]. Rosenthal [2006] may be a more elementary treatment.

- More importantly, the Lebesgue integral leads to limit theorems that do not hold for the Riemann integral. For example, one can prove sharper versions of central theorems in probability such as the Law of Large Numbers and the Central Limit Theorem. However, proving these theorems in their most abstract forms is of little interest to mathematicians in industry or to engineers.

- Rosenthal [2006, 1] motivates the Lebesgue integral by considering a mixed random variable (one with both discrete and continuous components). Specifically, he considers a Poisson variate and a normal variate and chooses one or the other based on a coin flip. The problem with this example is that it is easy to analyze this mixed variate without using measure theory.

A fellow I knew with a Ph.D. in analysis insisted that the Riemann integral was all one ever needs in industry. I find it hard to argue against that position. My contention, though, is that the Riemann integral is not taught to undergraduates until they take analysis. What they learn in calculus is that if $F(t)$ is the antiderivative of $f(t)$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

This result is what Yee and Výborný [2000, p. 1] and RA (p. 40) call *Newton's integral.* Calculus students do exercises related to the fundamental theorem of calculus, but that does not mean that they understand it. Most first-semester calculus students have difficulty viewing an expression such as $\int_a^x f(t)\,dt$ as a function of x. In any case, when do engineers and physicists or mathematicians working in industry pull out the definition of the Riemann integral? They generally use just Newton's integral, and the closest they get to the Riemann integral is numerical integration. From their vantage point, the Lebesgue integral is more abstract than the Riemann integral, is a great deal more complex, and has little utility.

The integral of Denjoy and Perron is more general than the Lebesgue integral and definitely more abstract. In the 1950s, Kurzweil and Henstock came up with a generalization of the Riemann integral that turns out to be equivalent to that integral. This integral is known variously as the *Kurzweil integral,* the *Henstock integral,* the *gauge integral,* or the *K–H integral;* I will refer to it as the *generalized Riemann integral.*

Yee and Výborný [2000] offer a worthwhile introduction to the generalized Riemann integral, as does Swartz [2001]; but the classic work by McLeod [1980] is the gold standard.

Amazingly, the generalized Riemann integral is barely more abstract conceptually than the Riemann integral and can be defined with almost exactly the same definition. That is, you can take a definition of the Riemann integral and slightly augment the wording to get a definition of the generalized Riemann integral. A short and clear statement by the late Robert Bartle and five other mathematicians defines the generalized Riemann integral and discusses bringing this definition into basic calculus texts [Bartle et al. 1996]. Although I am skeptical about that goal, I am impressed by their argument. Certainly, the generalized Riemann integral can be brought into the undergraduate real analysis course; the graduate course can then be built around it. In so doing, we do not lose measure theory as such, but the measure of a set S is now defined as $\int_S 1$. (If the integral does not exist, then S is a nonmeasurable set.)

A Little History

Whether it is a good idea to take a historical approach to first learning a subject has two answers:

- A student should do whatever he or she finds helpful regardless of what others think.
- Some disciplines have nice historically oriented introductory texts, while other disciplines do not. For example, in number theory the text by Ore [1988] works very well as an introduction, whereas the text by Goldman [1997] does the same thing but at a higher level of mathematical maturity and covers a great deal more material. In analysis, David Bressoud has written two very well-reviewed historical texts on analysis [2006; 2008], with the second devoted to the history of the Lebesgue integral; Steven Krantz gave it a rave review in this journal [2008]. I like Dunham [2008] a great deal for a superb introduction to the Lebesgue integral. Both Bressoud and Dunham owe something to Hawkins [2001]. Both ERA and RA do a good job of integrating history into the text, although not on the scale of these books.

I believe that both ERA and RA are great additions to the literature on analysis. The first is a good investment for both undergraduates and graduate students studying analysis, the second is worthwhile for students studying the Lebesgue integral. They are well-written and rich in content.

Nevertheless, it is time for the mathematics community to switch to the generalized Riemann integral—and to develop the Lebesgue integral as needed from there.

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